同调代数与几何应用

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  成绩：30% 考勤

  30% 作业

  - 周后课堂提交作业.

  40% 口试.

- Cohomology of groups.

Ref K. Brown Cohomology of grps Springer 1982.

- History.

Def (Hurewitz 1936) A space X is called aspherical if it is path-connected and $\pi_i(X) = 0$ for all $i \geq 2$.

Thm (Hurewitz) If X and Y are aspherical space with the same fundamental grp then X and Y are homotopy equivalent.

two maps $f_0, f_1: X \to Y$ are homotopy if there exists a map $F: X \times [0,1] \to Y$ s.t.

$$F|_{X \times \{0\}} = f_0 \quad \text{and} \quad F|_{X \times \{1\}} = f_1$$

two space X, Y are called homotopy equivalent if there exist maps $f: X \to Y$ and $g: Y \to X$ s.t.

$$f \circ g \simeq id_Y \quad \text{and} \quad g \circ f \simeq id_X.$$

<u>Def</u>: $H_*(G) = H_*(K(G,1))$ where $K(G,1)$ is an aspherical space with fundamental gp

<u>observation</u>

$$H_0(G) = \mathbb{Z}.$$

$$H_1(G) = G/[G,G].$$

<u>problem</u> it is hard to calculate $H_n(G)$ for $n \geq 2$ in general.

Remark: Hopf has a description of $H_2(G)$ using a presentation of G.

[1940s) Algebraic definition of $H_*(G) / H^*(G)$.

Applications: ① Calculation of $K_*(R)$.

Borel. $K_*(\mathbb{Z}) \otimes \mathbb{Q}$

Quillen $K_*(\mathbb{F})$ where $\mathbb{F}$ is a finite field.

② Mumford Conjecture (1983).

2002

(Madsen-Weiss) Stable homology of $Map(\Sigma_{g,1})$

[2006. ICM plenary talk)

③ Finiteness properties.

1957

[Eilenberg-Ganea conjecture] If $gd\, G = 2$, does $cd\, G = 2$.

1941

[Whitehead conjecture] Let X be an aspherical CW complex. Is any subcomplex of X also aspherical

1997
[Bestvina-Brady] at least one of these two conjecture fails.

0. Some homological algebra.

Def: Let R be a ring with unit. A (left) R-module is an abelian grp M together with a left action $(r, x) \to rx$ of R on M s.t.

(1) $r(sx) = (rs)x$ and

(2) $(r+s)x = rx + sx$, $r(x+y) = rx + ry$

for all $r \in R$, $x, y \in M$.

(3) $1 \cdot x = x$ for all $x \in M$.

Exples: Every abelian grp is a $\mathbb{Z}$-module in which nx is the usual integer multiple, $nx = \underbrace{x + \cdots + x}_{n}$ when $n > 0$.

Def: Let M be a left R-module. A subset S of M is linearly indep. (over R) when $\sum_{s\in S} r_s s = 0$ with $r_s \in R$ for all s and $r_s = 0$ for all but finit s, implies $r_s = 0$ for all $s \in S$.

Def: A basis of a left R-module M is a linearly independent of M that generates (spans) M. A module is free when it has a basis, and it then free on that basis.

Remark/Exe. M is free if and only if $M \cong \bigoplus_{i\in I} R$ for set I.

Def: A left R-module P is projective if $\varphi : P \to N$, $\rho : M \to N$ are homos and ρ is surjective, then there is $\psi : P \to M$ s.t. $\varphi = \rho \circ \psi$. ie the following diagram commutes.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\psi"'] \arrow[d, "\varphi"] & \\
M \arrow[r, "\rho"'] & N \arrow[r] & 0
\end{tikzcd}$$

Rem/Exe: every free module is projective

Def: (1) A graded R-module is a sequence $C = (C_n)_{n\in\mathbb{Z}}$ of R-modules. If $x \in C_n$, then we say that x has degree n and write $\deg x = n$.

(2) a map of degree p from a graded R-module C to a graded R-module C' is a family of maps $f = (f_n : C_n \to C'_{n+p})_{n\in\mathbb{Z}}$ of R-module homomorphisms.

[$\deg(f(x)) = \deg f + \deg x$.]

(3) A chain complex over R is a pair (C, d) where C is a graded R-module and $d: C \to C$ is a degree -1 s.t. $d^2 = 0$

$$\cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \to \cdots C_1 \xrightarrow{d_1} C_0 \to \cdots$$

The map $d: C \to C$ is called the differential or boundary operator of C.

[we often suppress d from the notation and simply say that C is a chain complex]

(4) given a chain complex (C, d) we define

the cycles $Z(C) = \ker(d)$

boundaries $B(C) = \mathrm{im}(d)$

and homology $H(C) = Z(C)/B(C)$ $\quad H_n(C) = \ker d_n / \mathrm{im}\, d_{n+1}$

Example:

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow[n \to 2n]{d_3} \mathbb{Z} \xrightarrow[n \to \bar{n}]{d_2} \mathbb{Z}/2 \xrightarrow[\bar{n} \to 0]{d_1} \mathbb{Z} \to \cdots$$

$Z(C)$	$\cdots$	0	0	$2\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
$B(C)$	$\cdots$	0	0	$2\mathbb{Z}$	$\mathbb{Z}/2$	0	$0 \cdots$
$H(C)$		0	0	0	0	$\mathbb{Z}$	0

Def: C is called exact at C_n if $H_n(C) = 0$

i.e. $\ker d_n = \mathrm{im}\, d_{n+1}$.

$$\cdots \to C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \to \cdots$$

(5) when we have a graded modules C with an endomorphism d of square zero such that d has degree $+1$ instead of -1. In this case we write $C = (C^n)_{n\in\mathbb{Z}}$ and $d^n : C^n \to C^{n+1}$. Such a pair (C, d) is called a cochain complex.

Similarly, we define

Cocycles : $Z(C) = \ker(d)$

Coboundaries : $B(C) = \operatorname{Im}(d)$

Cohomology : $H(C) = (H^n(C))_{n\in\mathbb{Z}} := \ker d^n / \operatorname{Im} d^{n-1}$

(6) If (C,d) and (C',d') are chain complexes, then a chain map from C to C' is a graded module homomorphism $f : C \to C'$ of degree 0 s.t. $d'f = fd$. A homotopy h from a chain map f to a chain map g is a graded homomorphism $h : C \to C'$ of degree 1 s.t. $d'h + hd = f - g$.

We write $f \simeq g$ and say that f is homotopic to g if there is a homotopy from f to g.

prop: A chain map $f: C \to C'$ induces a map ① $H(f): H(C) \to H(C')$ and ② $H(f) = H(g)$ if $f \simeq g$.

proof: ① we have.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f"] & C_n \arrow[r, "d_n"] \arrow[d, "f"] & C_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\ \cdots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

$$H_n(C) = \frac{\ker(d_n)}{\operatorname{Im}(d_{n+1})}$$

to show $H_n(C) \to H_n(C')$ induced from $f: C_n \to C_n'$ is well defined, it suffices to show that

(1) $\forall x \in C_n$, if $d_n(x) = 0$. $d_n' \circ f(x) = 0$

(2) $\forall x \in C_{n+1}$, $\exists y \in C'_{n+1}$ s.t.

$d'_{n+1}(y) = f \circ d_{n+1}$ (choose $y = f(x)$).

② let $[x] \in H^n(C)$.

then $f([x])$ $g([x]) \in H^n(C')$ we have $h: C \to C'$ of degree 1 s.t. $f([x]) - g([x]) = \underbrace{d_{n+1} \circ h_n(x)}_{\in \operatorname{Im} d_{n+1}} + h_{n-1} \circ \underbrace{d_n(x)}_{x \in \ker d_n}$

$= 0 + 0 = 0$

Def: The abelian group of homotopy classes of chain maps $C \to C'$ will denoted by $[C, C']$

Def: let M, N be two left R-modules. The Set

$$\mathrm{Hom}_R(A, B) = \{f \mid f : A \to B\}$$

of all R-module homomorphisms f of A into B is an abelian group, under the addition defined for $f, g : A \to B$ by

$$(f+g)a = f(a) + g(a).$$

Def: let C, C' be two chain complexes. we can define another chain complex $\mathrm{Hom}_R(C, C')$:

$$\mathrm{Hom}_R(C, C')_n = \prod_{q \in \mathbb{Z}} \mathrm{Hom}_R(C_q, C'_{q+n}),$$

$D_n : \mathrm{Hom}_R(C, C')_n \to \mathrm{Hom}_R(C, C')_{n-1}$ is defined by

$$D_n(f) = d'f - (-1)^n f d.$$

Lemma: The 0-cycles are precisely the chain maps $C \to C'$, and the 0-boundaries are the null-homotopic chain maps.

Thus $H_0(\mathrm{Hom}_R(C, C')) = [C, C']$

proof: $D_0(f) = d'f - fd = 0$, where $f \in \prod_{q \in \mathbb{Z}} \mathrm{Hom}_R(C_q, C'_q)$

Thus f is a chain map.

The 0-boundarys are image of D_1:

where $D_1(f) = d'f + fd$,

$$f \in \prod_{q\in\mathbb{Z}} \mathrm{Hom}(C_q, C'_{q+1})$$

Thus f is a chain homotopy from $D_1(f)$ to 0

This means the 0-boundaries are the null-homotopic chain maps.

$\square$

More generally, there is an interpretation of $H_n(\mathrm{Hom}_R(C,C'))$ in terms of chain maps. Consider the chain complex $(\Sigma^n C, \Sigma^n d)$ defined by $(\Sigma^n C)_p = C_{p-n}$ $\Sigma^n d_p = (-1)^n d_{p-n}$

This complex is called the n-fold suspension of C.

[If $n=1$, we write ΣC instead of $\Sigma^1 C$). let $[C,C']_n = [\Sigma^n C, C']$

Then we have $H_n(\mathrm{Hom}_R(C,C')) = [C,C']_n$.

The elements of $[\ ,\ ]_n$ are called homotopy class of chain maps of deg n.

A chain map $f: C \to C'$ is called a homotopy equivalence if there is a chain map $f': C' \to C$ s.t. $f'f \simeq id_C$ and $ff' \simeq id_{C'}$.

And a chain map is called a weak equivalence if $H(f): H(C) \to H(C')$ is an isomorphism.

Prop: Any homotopy equivalence is a weak equivalence.

Def A chain complex is called contractible if it homotopy equivalent to the zero complex, i.e. $id_C \simeq 0$. A homotopy from id_C to 0 is called a contracting homotopy. C is called acyclic if $H_*(C) = 0$.