

RESEARCH OVERVIEW, 2022–2026

XIAOLEI WU

My research focuses on using topological methods to study problems in group theory. From 2022 to 2026, my work has explored several topics in geometric group theory, including the Boone–Higman conjecture, finiteness properties of groups, and the homology of big mapping class groups. Below are some highlights of my research during this period.

- (1) **Embedding results for groups.** In joint work with Kai-Uwe Bux and Claudio Llosa Isenrich [BLIW24], we proved the Boone–Higman conjecture for many classes of groups, including Baumslag–Solitar groups, finitely generated free-by-cyclic groups, and fundamental groups of compact virtually fibered 3-manifolds. In joint work with Martin Palmer [PW25b], we showed that every group of type F_n embeds into an acyclic group of type F_n that has no proper finite index subgroup. This extends the classical acyclic-embedding results of Kan–Thurston [KT76] ($n = 0$), Baumslag–Dyer–Heller [BDH80b] ($n = 1$), and Baumslag–Dyer–Miller [BDM83] ($n = 2$), as well as a more recent result of Bridson [Bri20] ($n = 2$). Our results also produce, for each $n \geq 2$, the first known examples of acyclic groups of type F_n but not F_{n+1} . These examples can moreover be chosen to be simple.
- (2) **Homological finiteness properties of simple groups.** In joint work with Claudio Llosa Isenrich and Eduard Schesler [LISW25], we showed that for every $n \geq 1$, there is a simple group of type FP_n that is neither finitely presented nor of type FP_{n+1} . In fact, we proved a simple-group analogue of the Bestvina–Brady theorem [BB97]; see Theorem 2.2.
- (3) **Homology of big mapping class groups.** Together with Martin Palmer, we initiated a systematic investigation of the homology of big mapping class groups. We computed, for example, the integral homology of the mapping class group of the plane minus a Cantor set which answered a question of Calegari and Chen, proved uncountability results in every positive degree for many big mapping class groups, and determined when homology classes can be supported on finite-type subsurfaces for a large family of big mapping class groups [PW24a, PW24b, PW25a].
- (4) **Group splittings and virtual properties.** With Shengkui Ye, we completed the classification of triangle Artin groups that split as graphs of finite-rank free groups [WY25b]; this, in particular, answered two questions of Jankiewicz [Jan24a, Question 2.2, Question 2.3] in the negative. Based on this, we also proved that most triangle Artin groups are virtually poly-free, partially answering a question of Bestvina [Bes99]. In our recent work [WY26], we proved that pure braid groups are RFRS and hence that braid groups are virtually RFRS. This answers a question posed by Agol in his ICM survey [Ago14, Question 11].

In the rest of this article, I describe these results and the methods behind them in more detail.

1. EMBEDDING RESULTS FOR GROUPS

Much like Whitney’s embedding theorem in manifold theory, embedding theorems in group theory often provide a powerful means of constructing groups with prescribed properties while preserving information about a given subgroup. The landmark result of Higman says that a finitely generated group embeds into a finitely presented group if and only if it is recursively presented.

1.1. The Boone–Higman conjecture and tree actions. The Boone–Higman conjecture, formulated in 1974 [BH74], states the following:

Conjecture 1.1. *A finitely generated group has solvable word problem if and only if it embeds in a finitely presented simple group.*

The conjecture also appears in the classical text of Lyndon and Schupp [LS77, p. 191]. The reverse implication is first observed by Kuznetsov [Kuz58], but the embedding direction is far from understood. A major difficulty is the scarcity of constructions that simultaneously guarantee finite presentability and simplicity.

With Bux and Llosa Isenrich [BLIW24], we developed a geometric method for proving the conjecture for groups acting on locally finite trees. The main theorem says the following:

Theorem 1.2. *Let G be a group acting faithfully and cocompactly on a locally finite tree. If all edge stabilizers for the action are finitely presented, then G embeds in a finitely presented simple group.*

The central new object in the proof is the *rigid permutation group* $\text{RP}_G(T)$. It consists of permutations of the vertices of T that, outside a finite subtree, agree on each complementary component with the action of some element of G . Thus $\text{RP}_G(T)$ contains both G and all compactly supported vertex permutations, while its induced action on the boundary is the topological full group of the boundary action. The enlarged action on the vertex set is oligomorphic, making it suitable for a twisted Brin–Thompson construction. The hard part then is to determine the finiteness properties of the corresponding twisted Brin–Thompsons.

For applications, the paper introduces generalized Baumslag–Solitar groups over a group A : finite graphs of groups whose vertex and edge groups are commensurable with A , with finite-index edge inclusions. A transfer principle shows that one faithful Bass–Serre action in a commensurability class suffices to treat every group in that class. This allows us to prove the Boone–Higman conjecture for many classes of groups.

Corollary 1.3. *The Boone–Higman Conjecture holds for the following classes of groups:*

- (1) *Baumslag–Solitar groups;*
- (2) *finitely generated free-by-cyclic groups;*
- (3) *all fundamental groups of fibered 3-manifolds;*
- (4) *Leary–Minasyan groups.*
- (5) *Euclidean triangle Artin groups.*

The first two cases had been posed explicitly in [BBMZ25, Problem 5.3].

1.2. Acyclic and boundedly acyclic groups. A group G is *acyclic* if $H_i(G; \mathbb{Z}) = 0$ for every $i > 0$, and is *boundedly acyclic* if $H_b^i(G; \mathbb{R}) = 0$ for every $i > 0$. These are related but rather different vanishing conditions. My work asks whether they can be imposed in the host groups while retaining properties such as finiteness properties. Recall that a group is of type F_n if it admits a classifying space with finite n -skeleton, and is of type F_∞ if it is of type F_n for every n . In particular, F_1 means finite generation and F_2 means finite presentability.

Boundedly acyclic embeddings. Given a group G , one can define the so called labelled Thompson group $V(G)$ so that G naturally sits inside $V(G)$. The construction of $V(G)$ goes back to Thompson’s paper [Tho80]. In joint work with my students Fan Wu, Mengfei Zhao, and Zixiang Zhou [WWZZ25], we build the fundamental properties of $V(G)$.

Theorem 1.4. *The natural map*

$$\iota_G: G \hookrightarrow V(G)$$

is a quasi-isometric embedding whenever G is finitely generated, and

$$V(G) \text{ is of type } F_n \iff G \text{ is of type } F_n.$$

Moreover, $V(G)$ is boundedly acyclic, 5-uniformly perfect, and has no proper finite-index subgroups.

The proof of boundedly acyclicity of $V(G)$ is inspired by that of Thompson group V [And22] which in turn is inspired by the recent breakthrough of Monod [Mon22]. Theorem 1.4 allows us to prove many new embedding results for groups; for example:

Corollary 1.5. *For every $n \geq 1$, every group of type F_n embeds quasi-isometrically into a boundedly acyclic group of type F_n with no nontrivial finite quotient.*

These results strengthen many earlier embedding theorems in complementary ways. Bridson constructed quasi-isometric embeddings into groups of the same finiteness type having no proper finite-index subgroups [Bri98], but without bounded acyclicity. Fournier-Facio, Löh, and Moraschini gave functorial finitely generated boundedly acyclic hosts and constructed the first finitely presented non-amenable boundedly acyclic example [FFLM24], but without the simultaneous control obtained here over finiteness length and finite quotients.

Acyclic embeddings. There is long-standing interest in acyclic groups. Notably, Mather used a clever trick in [Mat71a] to prove that the compactly-supported homeomorphism groups of finite-dimensional Euclidean spaces are acyclic. The idea of the proof leads to the definition of *mitotic groups* [BDH80a], *pseudo-mitotic groups* [Var85] and *binate groups* [Ber89], all of which are acyclic. A recent breakthrough of Szymik and Wahl [SW19] shows that the Thompson group V is acyclic. Their proof involves heavy algebro-topological machinery including homological stability, stable homotopy theory and algebraic K-theory.

In subsequent work with Palmer [PW25b], we proved that all labelled Thompson groups and all twisted Brin–Thompson groups are acyclic in ordinary integral homology. This gives the following theorems about acyclic groups:

Theorem 1.6. (1) *For every $n \geq 1$, every group of type F_n embeds quasi-isometrically into an acyclic group of type F_n having no proper finite-index subgroups.*
(2) *Every finitely generated group embeds quasi-isometrically into a two-generated simple acyclic group.*
(3) *For every $n \geq 1$, there is a simple acyclic group of type F_n but not F_{n+1} . For $n \geq 2$, these are the first known acyclic examples with this exact finiteness length, even without requiring simplicity.*

These theorems extend a classical sequence of results through all finiteness dimensions. Kan and Thurston embedded every group in an acyclic group of essentially the same cardinality [KT76]. Baumslag, Dyer, and Heller proved that every finitely generated group embeds in a finitely generated acyclic group [BDH80b]; Baumslag, Dyer, and Miller proved the finitely presented analogue [BDM83]. More recently, Bridson proved Theorem 1.6 for $n = 2$. Our proof builds on earlier work of Xin Li [Li25]. Basically, we construct topological groupoids so that their topological full groups are labelled Thompson groups or twisted Brin–Thompson groups. We then calculate the homology of these groups by calculating the homology of the corresponding groupoids via [Li25, Corollary D]. It is worth mentioning that our construction of these groupoids also find applications in the studying of topological groupoids [WZM25].

The final statement is especially interesting. A standard way to prove that a group of type F_n is not of type F_{n+1} is to find that homology in degree $n+1$ is not finitely generated; this strategy is unavailable for an acyclic group. Instead, the equivalence

$$V(G) \text{ is of type } F_k \iff G \text{ is of type } F_k$$

transfers the failure of F_{n+1} from the input using a quasi-retract argument while the additional global symmetry kills every positive-dimensional homology group. The simple examples arise from twisted Brin–Thompson groups associated with Houghton groups.

2. HOMOLOGICAL FINITENESS PROPERTIES OF SIMPLE GROUPS

Finiteness properties generalize finite generation and finite presentability to higher dimensions. Stallings [Sta63] constructed a group of type F_2 but not F_3 , and Bieri [Bie76] extended this to groups of type F_n but not F_{n+1} in every higher dimension.

Constructing simple groups with prescribed finiteness properties proved more difficult. Brown [Bro87] showed that Thompson’s simple groups T and V are of type F_∞ . Skipper, Witzel, and Zaremsky [SWZ19] subsequently constructed, for every $n \geq 1$, a simple group of type F_n but not $FP_{n+1}(\mathbb{Z})$, using Röver–Nekrashevych groups. For $n \geq 2$, these were the first finitely presented simple groups with this exact finiteness length. Belk and Zaremsky [BZ22] later obtained another family of simple groups of type F_n but not F_{n+1} through twisted Brin–Thompson groups.

A finer distinction arises from the homological finiteness condition $FP_n(R)$: the trivial RG -module R admits a projective resolution finitely generated through degree n . Type F_n implies type $FP_n(R)$, but the converse can fail. For finitely presented groups, however, F_n and $FP_n(\mathbb{Z})$ are equivalent. Separating these two conditions therefore requires infinitely presented groups. The fundamental examples were constructed by Bestvina and Brady [BB97].

Let Γ be a finite nonempty graph, let $L = \text{Flag}(\Gamma)$ be its flag complex, and let A_Γ be the associated right-angled Artin group. The *Bestvina–Brady group* is

$$BB_\Gamma = \ker(A_\Gamma \longrightarrow \mathbb{Z}),$$

where every standard generator maps to 1.

Theorem 2.1 (Bestvina–Brady [BB97]). *For every $n \geq 1$ and every nonzero commutative unital ring R ,*

$$BB_\Gamma \text{ is of type } F_n \iff L \text{ is } (n-1)\text{-connected},$$

$$BB_\Gamma \text{ is of type } FP_n(R) \iff \tilde{H}_i(L; R) = 0 \text{ for all } 0 \leq i < n.$$

Moreover, BB_Γ is of type F_∞ if and only if L is contractible, and is of type $FP_\infty(R)$ if and only if L is R -acyclic.

In joint work with Claudio Llosa Isenrich and Eduard Schesler [LISW25], we proved that Bestvina–Brady groups can be replaced by simple groups:

Theorem 2.2. [LISW25, Theorem 1.1] *For every finite nonempty graph Γ and every $n \geq 1$ and every nonzero commutative unital ring R , there is a simple group G_Γ such that*

$$G_\Gamma \text{ is of type } F_n \iff L \text{ is } (n-1)\text{-connected},$$

$$G_\Gamma \text{ is of type } FP_n(R) \iff \tilde{H}_i(L; R) = 0 \text{ for all } 0 \leq i < n.$$

Our proof uses Röver–Nekrashevych groups. Basically, we use the fact that BB_Γ is linear over $\mathbb{Z}$ to build a self-similar action of $\mathbb{Z}^n \rtimes BB_\Gamma$ on some rooted tree. By marrying this group with the Thompson group using the self-similar action, we obtain some Röver–Nekrashevych group. The remaining difficulty then is to manipulate the action so that we can achieve the required finiteness property, and the resulting Röver–Nekrashevych groups are virtually simple.

Choosing finite flag complexes with a nontrivial fundamental group and vanishing homology in suitable dimensions gives the following consequences.

Corollary 2.3 ([LISW25, Corollaries 1.2 and 1.3]). *(1) For every $n \geq 1$, there is a simple group of type $FP_n(\mathbb{Z})$ that is neither finitely presented nor of type $FP_{n+1}(\mathbb{Z})$.*

(2) There is a simple group of type $FP_\infty(\mathbb{Z})$ that is not finitely presented.

These results provide the first infinitely presented simple groups of type $FP_2(\mathbb{Z})$, including examples satisfying homological finiteness in every degree. Since our examples arise as Röver–Nekrashevych groups, they also answer Zaremsky’s question whether a Thompson-like group can be of type $FP_2(\mathbb{Z})$ without being finitely presented.

3. HOMOLOGY OF BIG MAPPING CLASS GROUPS

The homology of finite-type mapping class groups has been a central topic for decades [Har85, MW07], but until recently little was known about the higher homology of big mapping class groups, namely mapping class groups of infinite-type surfaces. Throughout, we consider group homology with integral coefficients, treating these groups as discrete groups. In three papers with Martin Palmer [PW24b, PW24a, PW25a], we initiated a systematic study of this problem. The results reveal three sharply different phenomena: homology may vanish completely; it may be uncountable in every positive degree; and even when it is large, its classes may be invisible on every finite-type subsurface.

3.1. Acyclic big mapping class groups. The first paper [PW24b] treats a family of highly self-similar *binary-tree surfaces*.

Theorem 3.1. *If $\mathcal{B}(\Sigma)$ is the one-holed surface obtained by arranging copies of a connected surface Σ along an infinite rooted binary tree, then*

$$\tilde{H}_*(\text{Map}(\mathcal{B}(\Sigma)); \mathbb{Z}) = 0.$$

Thus these big mapping class groups are acyclic.

In particular, the mapping class group of the disc minus a Cantor set is acyclic. The proof of Theorem 3.1 has two ingredients. First, we use a homological stability argument to show that the natural inclusion map from $\mathcal{B}(\Sigma)$ to the boundary sum $\mathcal{B}(\Sigma) \sharp \mathcal{B}(\Sigma)$ induces isomorphism on homology in every degree for the corresponding mapping class groups. Second, we develop a homological version of Mather’s suspension trick [Mat71b] to calculate the homology. The idea is inspired by the proof of Thompson group V is acyclic [SW19], but Mather’s trick makes our proof much easier.

Replacing the boundary of the disk by a puncture gives the complete the following short exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow (\text{Map}(D \setminus \mathcal{C}) \rightarrow (\text{Map}(\mathbb{R}^2 \setminus \mathcal{C}) \rightarrow 1$$

Apply now the Lyndon–Hochschild–Serre spectral sequence, we have:

Corollary 3.2.

$$H_i(\mathrm{Map}(\mathbb{R}^2 \setminus \mathcal{C}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i \text{ even}, \\ 0, & i \text{ odd}, \end{cases}$$

where $\mathcal{C}$ is a Cantor set.

This answers affirmatively Question A.15 of Calegari and Chen [CC21], which asked in particular whether the second integral cohomology is isomorphic to $\mathbb{Z}$.

3.2. Big mapping class groups with uncountable homology in every degree. The companion paper [PW24a] discovers the opposite extreme. Let S be any infinite-type surface, $\mathrm{PMap}_c(S)$ be the subgroup of $\mathrm{Map}(S)$ with compact support, $\overline{\mathrm{PMap}_c(S)}$ be the topological closure of $\mathrm{PMap}_c(S)$ and $\mathcal{T}(S)$ be its Torelli group. Then

Theorem 3.3. *For every $i \geq 1$,*

$$H_i(\overline{\mathrm{PMap}_c(S)}; \mathbb{Z}) \quad \text{and} \quad H_i(\mathcal{T}(S); \mathbb{Z})$$

are uncountable; indeed, they contain $\bigoplus_{\mathfrak{c}} \mathbb{Z}$, where $\mathfrak{c}$ is the cardinality of the continuum.

We also use this to prove that $\mathrm{Map}(S)$ is uncountable in all positive degrees for a large class of infinite-type surfaces. Here the proof again has two ingredients. First, it is known from Domat's work [Dom22] that $\mathrm{PMap}_c(S)$ and $\mathcal{T}(S)$ have $\overline{\mathrm{PMap}_c(S)}$ is uncountable in all positive degrees. Our idea then is to promote this to higher degrees using an exterior algebra argument.

3.3. Compact and finite-type support. For any infinite-type surface S , a natural question is whether the homology of its mapping class group contains any non-trivial classes that are supported on a compact or finite type subsurface. The question is interesting because there are already many calculations about the homology of mapping class groups of finite type surfaces. If these homology classes survive in the homology of the ambient big mapping class groups, we would obtain many new calculations about homology of the big mapping class groups. We provide an almost-complete answer when the genus of S is positive (including infinite) and a partial answer when the genus of S is zero. The most interesting case for our calculation is when S is an infinite finite genus surface without isolated planar ends.

Theorem 3.4. *Let S be an infinite genus surface without isolated planar ends. Then no nontrivial homology class of $H_*(\mathrm{Map}(S); \mathbb{Q})$ can be supported on a compact or finite type surface.*

For example, take S to be the Loch Ness monster surface; our theorem implies that all the dual Miller–Morita–Mumford classes are sent to 0 in $H_*(\mathrm{Map}(S); \mathbb{Q})$.

Our methods involve the notion of *shiftable subsurfaces* as well as homological stability for mapping class groups of finite-type surfaces. The key idea is, again, Mather's suspension trick.

4. GROUP SPLITTINGS AND VIRTUAL PROPERTIES

My work on Artin groups, free-by-cyclic groups, and tubular groups investigates how group splittings interact with geometric and residual properties. Using Bass–Serre theory, cubical geometry, and many other tools, my collaborators and I obtain classification results, construct counterexamples, and establish virtual properties for several families of groups.

4.1. Triangle Artin groups and actions on trees. For integers $M, N, P \geq 2$, the triangle Artin group is

$$\mathrm{Art}_{MNP} = \langle a, b, c \mid \{a, b\}_M = \{b, a\}_M, \{b, c\}_N = \{c, b\}_N, \{c, a\}_P = \{a, c\}_P \rangle,$$

where $\{x, y\}_r = xyx \cdots$ is the alternating word of length r beginning with x . A splitting as a graph of finite-rank free groups is, by Bass–Serre theory, equivalent to a nontrivial cocompact action on a simplicial tree whose vertex and edge stabilizers are finite-rank free groups. Jankiewicz proved that every large-type triangle Artin group (i.e. all labels are at least 3) admits such a splitting [Jan22], and subsequently treated the groups Art_{2MN} with $M, N \geq 4$ [Jan24b]. She was also able to use these splittings to show that many of them are residually finite. Together with Squier's earlier work on the Euclidean case [Squ87], this leads her to ask the following question.

Question 4.1. [Jan24b, Questions 2.2 and 2.3] *Does the Artin group $\mathrm{Art}_{2,3,M}$ where $M \geq 7$ split as a graph of finite rank free groups? How about all 2-dimensional Artin groups?*

In joint work with Shengkui Ye, we answered both questions in the negative and completed the classification for triangle Artin groups [WY25b].

Theorem 4.2.

Art_{23M} splits as a graph of finite-rank free groups $\iff M > 5$ is even.

Together with Squier's and Jankiewicz's results, this gives a complete classification: for $2 \leq M \leq N \leq P$, the group Art_{MNP} splits as a graph of finite-rank free groups if and only if

$$\frac{1}{M} + \frac{1}{N} + \frac{1}{P} \leq 1 \quad \text{and} \quad (M, N, P) \neq (2, 3, 2k+1) \quad (k \geq 3).$$

Note that the condition $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} \leq 1$ is the same as asking the Art_{MNP} to be 2-dimensional. The positive and negative parts require different ideas. For even M , a nonstandard presentation is realized by a presentation complex equipped with a height function; cutting along a regular level set produces the explicit amalgam $F_3 *_{F_7} F_4$. For odd M , we analyze minimal actions of $\text{Art}_{2,2n+1,2m+1}$ on simplicial trees. A trichotomy forces either a fixed point, an invariant line, or a stabilizer containing a prescribed nonfree subgroup. In every case, the behavior is incompatible with a graph-of-free-groups splitting.

The paper also advances Bestvina's question on virtual poly-freeness of Artin groups [Bes99]. Despite some progress has been made on this [BG21, BGMPP19, Wu22], it is still widely open for Artin groups in general.

Theorem 4.3. Art_{MNP} is virtually poly-free whenever the ordered label triple is not $(2, 3, 2k+1)$, $k \geq 3$.

A main ingredient is that an algebraically clean graph of free groups, in which every edge group embeds as a free factor of each adjacent vertex group, is normally poly-free.

4.2. Free-by-cyclic and tubular groups. In further work with Ye [WY25a], we study the interaction between free-by-cyclic groups and tubular groups.

Free-by-cyclic groups turns out to be more common than one would think. Kielak and Linton [KL24] answered an old question of Baumslag by showing that one relator groups with torsion are virtually free-by-cyclic. Fisher improves their criterion for showing free-by-cyclic by establishing: a finitely generated RFRS group G is virtually-free-by-cyclic if and only if its second ℓ^2 -Betti number vanishes and $\text{cd}_{\mathbb{Q}}(G) \leq 2$. [Fis26, Theorem A]. So it is reasonable to ask the following:

Question 4.4. [HW10a, Remark 3.6] *Is the amalgamated product of two free-by-cyclic groups over a cyclic subgroup (virtually) free-by-cyclic?*

We answer this question negatively using Gersten's group.

Theorem 4.5. Let $G_1 = \langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ba, tct^{-1} = ca^2 \rangle$ be the Gersten group, $G_2 = \langle x, s \mid sxs^{-1} = x \rangle \cong \mathbb{Z}^2$. Then the tubular group $G = G_1 *_{a=x} G_2$ is not virtually free-by-cyclic.

The key idea here is to view Gersten group as a tubular group and see that the generator a is distorted. Similar idea also allows us to prove the following:

Theorem 4.6. *The Gersten group*

$$\langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ba, tct^{-1} = ca^2 \rangle$$

is a cyclic-subgroup-separable tubular group that does not have the property (VRC).

This answers [Min21, Question 11.6] in the negative. Recall that a group G is (VRC) (virtual retract onto any cyclic subgroup) if, for every cyclic subgroup C , there is a finite index subgroup $H \leq G$ such that $H \geq C$ and H have an epimorphism $H \rightarrow C$ satisfying that $C \rightarrow H \rightarrow C$ is the identity. A subgroup K of a group G is called separable if K is the intersection of a family of finite index subgroups in G .

4.3. virtual RFRSness of Artin groups. Residual properties of groups play a central role in understanding their structure and applications to topology and geometry. Among these, the class of *residually finite rationally solvable* (RFRS) groups, introduced by Agol [Ago08], has been particularly influential due to its connections with virtual properties of 3-manifolds [Ago14] and fibering [Kie20]. Recall that a group G is called RFRS if there is a descending chain $G = G_0 \geq G_1 \geq G_2 \geq \dots$ of finite-index normal subgroups of G with trivial intersection, such that for each i the subgroup G_{i+1} contains the kernel of $G_i \rightarrow H_1(G_i; \mathbb{Q})$. Agol asks in [Ago14, Question 11] whether braid groups are RFRS. This cannot be true since RFRS groups are locally indicable [Fis24, Proposition 2.7], but the commutator subgroups of braid groups B_n are finitely generated and perfect once $n \geq 5$ [GL69, Theorem 2.1]. In [WY25a, §7], we modified his question to the following:

Question 4.7. *Are braid groups virtually RFRS?*

In a recent joint work with Shengkui Ye, we answer this in the positive.

Theorem 4.8. *The pure braid groups are RFRS.*

Our result also implies that many other Artin groups are virtually RFRS, including those of types $B_n = C_n$, $\tilde{C}_n$. These also provide evidence toward the problem of whether braid groups, and more generally Artin groups, are virtually special; see [HW10b, Problem 9.4], [Wis14, Problem 13.4].

Our proof uses finite covers of planar surfaces to detect nontrivial loops in first homology modulo p , together with a lifting argument for suitable finite-index subgroups of pure braid groups. One of the key observations is that the pure braid groups acting trivially on the first homology of the punctured disk. The Fadell–Neuwirth sequence [FN62] then allows us to prove the result by induction on the number of strands. The proof also uses an alternative definition of RFRS developed in my joint work with Xiaoming Huang and Ye [HWY26]. There we prove that (virtually) RFRSness and many other properties are closed under taking graph products.

REFERENCES

- [Ago08] Ian Agol. Criteria for virtual fibering. *J. Topol.*, 1(2):269–284, 2008.
- [Ago14] Ian Agol. Virtual properties of 3-manifolds. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. I*, pages 141–170. Kyung Moon Sa, Seoul, 2014.
- [And22] Konstantin Andritsch. Bounded cohomology of groups acting on cantor sets. [arXiv:2210.00459](https://arxiv.org/abs/2210.00459), 2022.
- [BB97] Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.
- [BBMZ25] James Belk, Collin Bleak, Francesco Matucci, and Matthew CB Zaremsky. Progress around the Boone–Higman conjecture. *EMS Surveys in Mathematical Sciences*, 2025. Online first.
- [BDH80a] G. Baumslag, E. Dyer, and A. Heller. The topology of discrete groups. *J. Pure Appl. Algebra*, 16(1):1–47, 1980.
- [BDH80b] Gilbert Baumslag, Eldon Dyer, and Alex Heller. The topology of discrete groups. *J. Pure Appl. Algebra*, 16(1):1–47, 1980.
- [BDM83] Gilbert Baumslag, Eldon Dyer, and III Miller, Charles F. On the integral homology of finitely presented groups. *Topology*, 22(1):27–46, 1983.
- [Ber89] A. J. Berrick. Universal groups, binate groups and acyclicity. In *Group theory (Singapore, 1987)*, pages 253–266. de Gruyter, Berlin, 1989.
- [Bes99] Mladen Bestvina. Non-positively curved aspects of Artin groups of finite type. *Geom. Topol.*, 3:269–302, 1999.
- [BG21] Rubén Blasco-García. Poly-freeness in large even Artin groups. *J. Pure Appl. Algebra*, 225(4):Paper No. 106551, 26, 2021.
- [BGMPP19] Rubén Blasco-García, Conchita Martínez-Pérez, and Luis Paris. Poly-freeness of even Artin groups of FC type. *Groups Geom. Dyn.*, 13(1):309–325, 2019.
- [BH74] William W. Boone and Graham Higman. An algebraic characterization of groups with soluble word problem. *J. Austral. Math. Soc.*, 18(1):41–53, 1974.
- [Bie76] Robert Bieri. Normal subgroups in duality groups and in groups of cohomological dimension 2. *J. Pure Appl. Algebra*, 7(1):35–51, 1976.
- [BLIW24] Kai-Uwe Bux, Claudio Llosa Isenrich, and Xiaolei Wu. On the Boone–Higman conjecture for groups acting on locally finite trees. *Duke Math. J.*, 2024. To appear.
- [Bri98] Martin R. Bridson. Controlled embeddings into groups that have no non-trivial finite quotients. In *The Epstein Birthday Schrift*, volume 1 of *Geom. Topol. Monogr.*, pages 99–116. Geom. Topol. Publ., Coventry, 1998.
- [Bri20] Martin R. Bridson. The homology of groups, profinite completions, and echoes of Gilbert Baumslag. In *Elementary theory of groups and group rings, and related topics*, De Gruyter Proc. Math., pages 11–27. De Gruyter, Berlin, 2020.
- [Bro87] Kenneth S. Brown. Finiteness properties of groups. In *Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985)*, volume 44, pages 45–75, 1987.
- [BZ22] James Belk and Matthew C. B. Zaremsky. Twisted Brin–Thompson groups. *Geom. Topol.*, 26(3):1189–1223, 2022.
- [CC21] Danny Calegari and Lvzhou Chen. Big mapping class groups and rigidity of the simple circle. *Ergodic Theory Dynam. Systems*, 41(7):1961–1987, 2021.
- [Dom22] George Domat. Big pure mapping class groups are never perfect. *Math. Res. Lett.*, 29(3):691–726, 2022. Appendix with Ryan Dickmann.
- [FFLM24] Francesco Fournier-Facio, Clara Löh, and Marco Moraschini. Bounded cohomology of finitely presented groups: vanishing, non-vanishing, and computability. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 25(2):1169–1202, 2024.
- [Fis24] Sam P. Fisher. Improved algebraic fibrings. *Compos. Math.*, 160(9):2203–2227, 2024.
- [Fis26] Sam P. Fisher. On the cohomological dimension of kernels of maps to $\mathbb{Z}$. *Geom. Topol.*, 30(1):373–388, 2026.
- [FN62] Edward R. Fadell and Lee Neuwirth. Configuration spaces. *Math. Scand.*, 10:111–118, 1962.
- [GL69] E. A. Gorin and V. Ja. Lin. Algebraic equations with continuous coefficients, and certain questions of the algebraic theory of braids. *Mat. Sb. (N.S.)*, 78(120):579–610, 1969.
- [Har85] John L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Ann. of Math. (2)*, 121(2):215–249, 1985.
- [HW10a] Mark Hagen and Daniel T. Wise. Special groups with an elementary hierarchy are virtually free-by- $\mathbb{Z}$. *Groups Geom. Dyn.*, 4(3):597–603, 2010.

- [HW10b] Frédéric Haglund and Daniel T. Wise. Coxeter groups are virtually special. *Adv. Math.*, 224(5):1890–1903, 2010.
- [HWY26] Xiaoming Huang, Xiaolei Wu, and Shengkui Ye. Virtual inheritance properties of graph products. [arXiv:2606.14148](#), 2026.
- [Jan22] Kasia Jankiewicz. Residual finiteness of certain 2-dimensional Artin groups. *Adv. Math.*, 405:Paper No. 108487, 2022. 37 pp.
- [Jan24a] Kasia Jankiewicz. Splittings of triangle Artin groups. *Groups Geom. Dyn.*, 18(1):91–108, 2024.
- [Jan24b] Kasia Jankiewicz. Splittings of triangle Artin groups. *Groups Geom. Dyn.*, 18(1):91–108, 2024.
- [Kie20] Dawid Kielak. Residually finite rationally solvable groups and virtual fibering. *J. Amer. Math. Soc.*, 33(2):451–486, 2020.
- [KL24] Dawid Kielak and Marco Linton. Virtually free-by-cyclic groups. *Geom. Funct. Anal.*, 34(5):1580–1608, 2024.
- [KT76] Daniel M. Kan and William P. Thurston. Every connected space has the homology of a $K(\pi, 1)$. *Topology*, 15(3):253–258, 1976.
- [Kuz58] Alexander Kuznetsov. Algorithms as operations in algebraic systems. In *Proceedings of the All-Union Colloquium on Algebra Article 2 from the morning session of 6th February 1958*, volume 13.3, pages 240–241, 1958.
- [Li25] Xin Li. Ample groupoids, topological full groups, algebraic K -theory spectra and infinite loop spaces. *Forum Math. Pi*, 13:Paper No. e9, 2025. 56 pp.
- [LISW25] Claudio Llosa Isenrich, Eduard Schesler, and Xiaolei Wu. Infinitely presented simple groups separated by homological finiteness properties. [arXiv:2510.01952](#), 2025.
- [LS77] Roger C. Lyndon and Paul E. Schupp. *Combinatorial Group Theory*, volume 89 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin–New York, 1977.
- [Mat71a] John N. Mather. The vanishing of the homology of certain groups of homeomorphisms. *Topology*, 10:297–298, 1971.
- [Mat71b] John N. Mather. The vanishing of the homology of certain groups of homeomorphisms. *Topology*, 10(4):297–298, 1971.
- [Min21] Ashot Minasyan. Virtual retraction properties in groups. *Int. Math. Res. Not. IMRN*, (17):13434–13477, 2021.
- [Mon22] Nicolas Monod. Lamplighters and the bounded cohomology of Thompson’s group. *Geom. Funct. Anal.*, 32(3):662–675, 2022.
- [MW07] Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford’s conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007.
- [PW24a] Martin Palmer and Xiaolei Wu. Big mapping class groups with uncountable integral homology. *Doc. Math.*, 29(1):159–189, 2024.
- [PW24b] Martin Palmer and Xiaolei Wu. On the homology of big mapping class groups. *J. Topol.*, 17(4):Paper No. e12358, 2024. 41 pp.
- [PW25a] Martin Palmer and Xiaolei Wu. Compact and finite-type support in the homology of big mapping class groups. *J. Lond. Math. Soc. (2)*, 112(3):Paper No. e70258, 2025. 35 pp.
- [PW25b] Martin Palmer and Xiaolei Wu. Embedding groups into acyclic groups. [arXiv:2510.16879](#), 2025.
- [Squ87] Craig C. Squier. On certain 3-generator Artin groups. *Trans. Amer. Math. Soc.*, 302(1):117–124, 1987.
- [Sta63] John Stallings. A finitely presented group whose 3-dimensional integral homology is not finitely generated. *Amer. J. Math.*, 85:541–543, 1963.
- [SW19] Markus Szymik and Nathalie Wahl. The homology of the Higman–Thompson groups. *Invent. Math.*, 216(2):445–518, 2019.
- [SWZ19] Rachel Skipper, Stefan Witzel, and Matthew C. B. Zaremsky. Simple groups separated by finiteness properties. *Invent. Math.*, 215(2):713–740, 2019.
- [Tho80] Richard J. Thompson. Embeddings into finitely generated simple groups which preserve the word problem. In *Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976)*, volume 95 of *Stud. Logic Found. Math.*, pages 401–441. North-Holland, Amsterdam–New York, 1980.
- [Var85] K. Varadarajan. Pseudo-mitotic groups. *J. Pure Appl. Algebra*, 37(2):205–213, 1985.
- [Wis14] Daniel T. Wise. The cubical route to understanding groups. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1075–1099. Kyung Moon Sa, Seoul, 2014.
- [Wu22] Xiaolei Wu. Poly-freeness of Artin groups and the Farrell–Jones conjecture. *J. Group Theory*, 25(1):11–24, 2022.
- [WWZZ25] Fan Wu, Xiaolei Wu, Mengfei Zhao, and Zixiang Zhou. Embedding groups into boundedly acyclic groups. *J. Lond. Math. Soc. (2)*, 111(5):Paper No. e70164, 2025.
- [WY25a] Xiaolei Wu and Shengkui Ye. Some questions related to free-by-cyclic groups and tubular groups. [arXiv:2504.14192](#), 2025.
- [WY25b] Xiaolei Wu and Shengkui Ye. Splittings and poly-freeness of triangle Artin groups. *Trans. Amer. Math. Soc.*, 378(7):4707–4737, 2025.
- [WY26] Xiaolei Wu and Shengkui Ye. Pure braid groups are RFRS. [arXiv:2608.13978](#), 2026.
- [WZM25] Xiaolei Wu, Mengfei Zhao, and Xin Ma. Ample groupoids that are neither almost finite nor purely infinite. [arXiv:2512.01325](#), 2025.

SHANGHAI CENTER FOR MATHEMATICAL SCIENCES, JIANGWAN CAMPUS, FUDAN UNIVERSITY, NO. 2005 SONGHU ROAD, SHANGHAI 200438, P.R. CHINA
 Email address: xiaolei.wu@fudan.edu.cn